THE BINARY NUMBER SYSTEM

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Our civilization uses the **base 10** or **decimal** place value system. Each digit in a number represents a power of 10. For example, 365.42 means

$$3*10^2 + 6*10^1 + 5*10^0 + 4*10^{-1} + 2*10^{-2}$$

In general, think of the place values as powers of 10.

$$\cdots\ \begin{array}{cccccc} \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad}\ . & \underline{\quad} & \underline{\quad} \\ 10^3 & 10^2 & 10^1 & 10^0 & 10^{-1} & 10^{-2} \end{array}\ \cdots$$

There is nothing sacred about base 10. We could just as well use **base 2**, also called **binary**.

$$\cdots\ \begin{array}{cccccc} \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad}\ . & \underline{\quad} & \underline{\quad} \\ 2^3 & 2^2 & 2^1 & 2^0 & 2^{-1} & 2^{-2} \end{array}\ \cdots$$

For example, 1101.1 is a binary number. It means

$$1*2^3 + 1*2^2 + 0*2^2 + 1*2^1 + 1*2^{-1} = 8 + 4 + 1 + ½ = 13 ½ \text{ (base 10)}$$

That is, $1101.1_2 = 13.5_{10}$, where the subscript indicates the base.

In general, to convert binary to decimal, just expand out using place values. Try converting 1011010_2 to base 10. Did you get 90?

To convert a base 10 integer to binary, think of filling in blanks in the binary place value chart, starting as far left as possible. That is, pull out the largest power of 2 possible from the integer. Subtract it from the original number, and repeat with the remainder. Continue until the remainder is 0 or 1.

Example: Convert 85_{10} to binary.

Solution: 64 ($= 2^6$) is the largest power of 2 that is less than or equal to 85. Put 1 in the 64's place and subtract it off.

$$\begin{array}{ccccccc} \underline{1} & \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ 2^6 & 2^5 & 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \end{array} \qquad\qquad 85 - 64 = 21$$

Repeat, this time using 21. The largest power of 2 less than or equal to 21 is 16 (=2^3). Put 1 in the 16's position and subtract it off. Note that this also requires us to put 0 in the 32's position!

$$\begin{array}{ccccccc} \underline{1} & \underline{0} & \underline{1} & \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ 2^6 & 2^5 & 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \end{array} \qquad 21 - 16 = 5$$

Continuing, $5 - 4 = 1$. Put 1 in the 4's position (and 0 in the 8's position).

$$\begin{array}{ccccccc} \underline{1} & \underline{0} & \underline{1} & \underline{0} & \underline{1} & \underline{\quad} & \underline{\quad} \\ 2^6 & 2^5 & 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \end{array} \qquad 5 - 4 = 1$$

Since the remainder is 1, put 1 in the 1's position (and 0 in the 2's position).

$$\begin{array}{ccccccc} \underline{1} & \underline{0} & \underline{1} & \underline{0} & \underline{1} & \underline{0} & \underline{1} \\ 2^6 & 2^5 & 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \end{array}$$

That is, $85_{10} = 1010101_2$.

Try counting in binary, starting with 0. Do you get 0, 1, 10, 11, 100, 101, 110, 111, 1000, …?

To convert a base 10 fraction to binary, assume that the denominator is a power of 2. Other fractions are beyond the scope of these notes.

With that assumption, write the integer numerator in binary, write the denominator as a power of 2, and shift the binary point left the number of places indicated by the exponent.

Example: Convert $\frac{5}{8}$ to binary.

Solution: First, $5_{10} = 4 + 1 = 101_2$. Since $8 = 2^3$, shift the binary point three places to the left. $\frac{5}{8} = \frac{5}{2^3} = .101_2$

Convert a mixed base 10 number to base 2 by writing the mixed number as an improper fraction and proceeding as above.

Example: Convert 17 ¼ to base 2.

Solution: 17 ¼ = 69/4 . Now $69 = 64 + 4 + 1$, so $69_{10} = 1000101_2$. Next, since $4 = 2^2$, move the binary point two places to the left. Thus 17 ¼ = 10001.01_2 .

BASE 16 (HEXADECIMAL) NUMBERS

Just as we use base 10 and base 2 place values, we could use other bases. In particular, **base 16**, also called **hexadecimal**, is used frequently in computer science.

$$\cdots \underset{16^3}{\_\_} \quad \underset{16^2}{\_\_} \quad \underset{16^1}{\_\_} \quad \underset{16^0}{\_\_} \,.\, \underset{16^{-1}}{\_\_} \quad \underset{16^{-2}}{\_\_} \cdots$$

Since $16^2 = 256$, $16^3 = 4096$, and so on, hexadecimal place values increase rapidly!

Convert between base 10 and base 16 as we did earlier with binary.

Example: Convert 158_{16} to decimal.

Solution: $158_{16} = 1*16^2 + 5*16 + 8$

$= 256 + 80 + 8$

$= 344_{10}$

Example: Convert 643_{10} to base 16.

Solution: $643_{10} = 2*256 + 131$

$= 2*256 + 8*16 + 3$

$= 2*16^2 + 8*16^1 + 3*16^0 = 283_{16}$

A problem arises when we try to count in hexadecimal. What comes after 9?

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, ?

What should be the tenth positive integer? It cannot be written as 10, because

$10_{16} = 1*16 + 0 = 16_{10}$

We must have another symbol for the tenth digit. A little thought should convince you that we need symbols for the eleventh through the fifteenth digit, too. This will give 16 digits in all, including 0. As a general principle, base n requires n digits, including 0. Each digit must be a single symbol. The common agreement in mathematics is to use the letters *A, B, C, D, E,* and *F* as the new symbols.

Base 2 digits: 0, 1 (A **bit** is a binary digit)

Base 10 digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

Base 8 digits: 0, 1, 2, 3, 4, 5, 6, 7

Base 16 digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, *A*, *B*, *C*, *D*, *E*, *F*

Example: Convert $10A3_{16}$ to base 10.

Solution: $10A3_{16} = 1*16^3 + 0*16^2 + A*16^1 + 3*16^0$

$= 1*4096 + 10*16 + 3*1$

$= 4096 + 160 + 3 = 4259_{10}$

Use the extended set of digits as needed to convert decimal to base 16.

Example: Convert 445_{10} to hexadecimal.

Solution: $445_{10} = 1*256 + 189$

$= 1*256 + 11*16 + 13$

$= 1*16^2 + B*16^1 + D*16^0 = 1BD_{16}$

Converting between base 16 and base 2 is particularly useful. You can do this by going through base 10, of course, but it is much quicker to convert directly. Since $16 = 2^4$, one hexadecimal digit is the equivalent of four binary digits. To convert from base 2 to base 16, just group the base 2 bits by fours, starting on the right. Convert each group of four bits to its equivalent digit in base 16.

Example: Convert 1011011001_2 to hexadecimal.

Solution: Group the bits by fours, starting on the right. For emphasis, write two leading zeros to round out the first group of bits.

1011011001 = 0010 1101 1001

Now $0010_2 = 2_{16}$, $1101_2 = 13_{10} = D_{16}$, and $1001_2 = 9_{16}$, so $1011011001_2 = 2D9_{16}$.

Example: Convert $B605_{16}$ to base 2.

Solution: Expand each hexadecimal digit to its four-bit binary form.

$B_{16} = 11_{10} = 1011_2$, $6_{16} = 0110_2$, $0_{16} = 0000_2$, $5_{16} = 0101_2$.

Therefore, $B605_{16} = 1011011000000101_2$