Homework 1

3rd of February, 2016

This homework is to be done in the groups you picked. Everyone in the group is responsible for knowing how to solve all of the problems. Your group needs to meet (as a group!) at least twice. At least two members should be present to get help on a homework problem.

Problems

1. Let $\mathbf{G}$ be a graph. Show that $\mathbf{G}$ and $\overline{\mathbf{G}}$ cannot both be disconnected.
2. Let $\mathbf{G}$ be a graph with two or more vertices. Prove that $\mathbf{G}$ must contain two or more vertices of the same degree.
3. Let $\mathbf{G}$ be a graph with n vertices and m edges and with an adjacency matrix $A = [a_{ij}]$. Let $A^2 = [b_{ij}]$ and $A^3 = [c_{ij}]$. (Tr() is the trace of the matrix – the sum of the diagonal)
   (a) Show that $b_{ii} = \deg(v_i)$ and that $\mathrm{Tr}(A^2) = 2m$
   (b) Show that the number of triangles in $\mathbf{G}$ containing v_i, is $\frac{1}{2}c_{ii}$ and that the total number of triangles in $\mathbf{G}$ is $\frac{1}{6}\mathrm{Tr}(A^3)$.
4. Using problem three and the Adjacency Matrix Theorem show that if A_n were the adjacency matrix of $\mathbf{K}_n$, then $A_n^2 = (n-2)J_n + I_n$. Where J_n is the matrix of all ones and I_n is the identity matrix.
5. Show that if $\mathbf{G}$ were self-complementary with n vertices, then either $n = 4k$ or $n = 4k+1$.
6. The **line graph** $\mathbf{L(G)}$ of a simple graph $\mathbf{G}$ is the graph whose *vertices* are in one-to-one correspondence with the *edges* of $\mathbf{G}$, two vertices of $\mathbf{L(G)}$ being adjacent if and only if the corresponding edges of $\mathbf{G}$ are adjacent.
   (a) Show that $\mathbf{K}_3$ and $\mathbf{K}_{1,3}$ have the same line graph.
   (b) Find an expression for the number of edges of $\mathbf{L(G)}$ in terms of the degrees of the vertices of $\mathbf{G}$.
   (c) Show that if $\mathbf{G}$ were regular of degree k, then $\mathbf{L(G)}$ would be regular of degree $2k-2$.
7. Draw all trees of size 6, 7, and 8.
8. Find and take a picture (with all group members) of four trees that are not trees. Your trees should be on the Longwood campus.