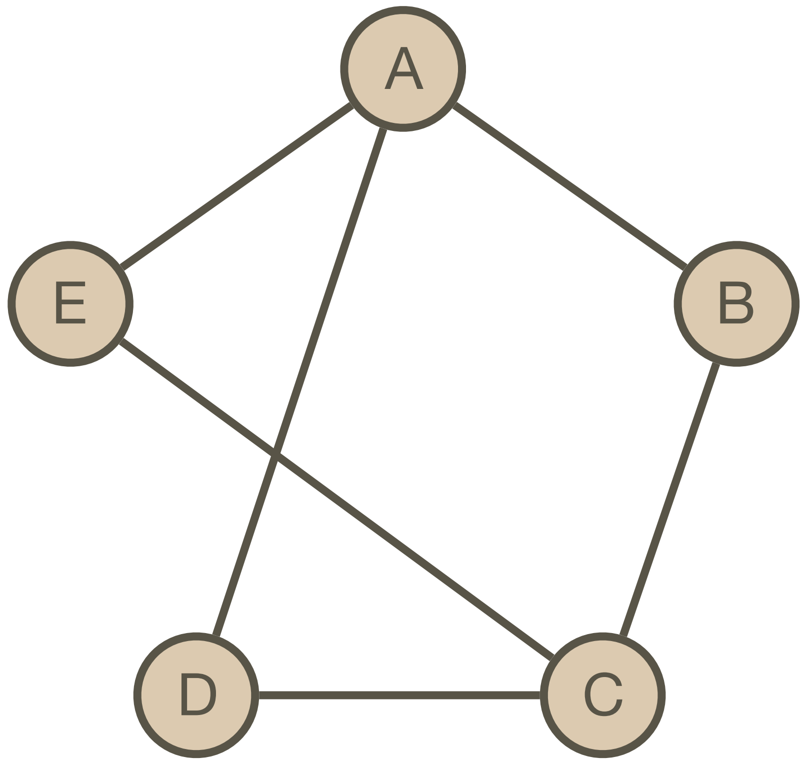
Adjacency Matrix Theorem, Connectivity, and Trees

CMSC/MATH 420

Adjacency Matrix Theorem

- Let **A** be an adjacency matrix for a graph **G** with *n* vertices. If $\mathbf{A}^k = [n_{ij}]$, then n_{ij} is the number of walks of length *k* from $\mathbf{v}_i$ to $\mathbf{v}_j$.

- Proof using the Principle of Mathematical Induction
  - Establish a base case
  - Assume the condition is true for the k case
  - Prove the condition is true for the k+1 case
- A lot like a computer science iteration/recursion

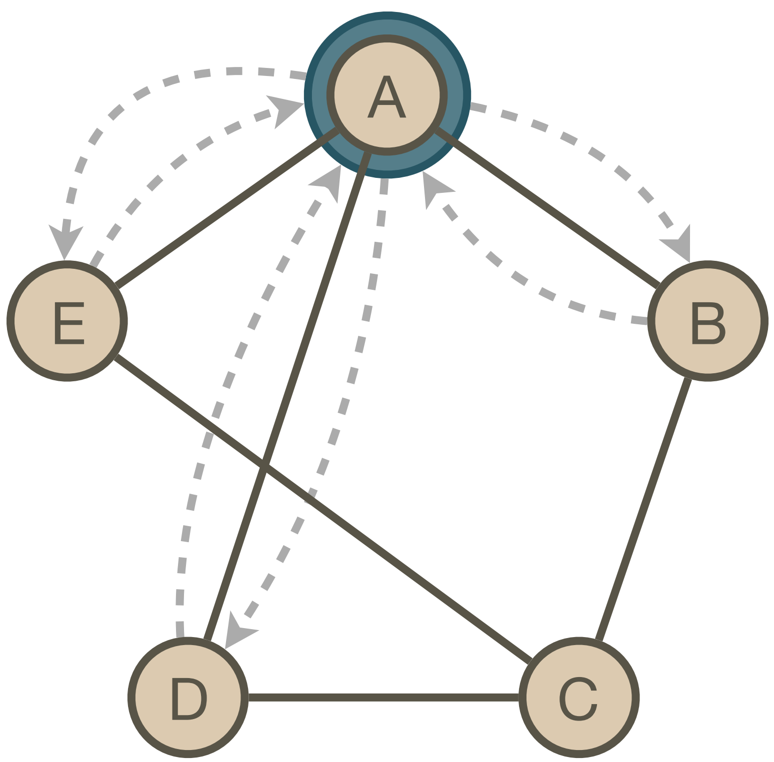


	A	B	C	D	E
A	0	1	0	1	1
B	1	0	1	0	0
C	0	1	0	1	1
D	1	0	1	0	0
E	1	0	1	0	0

The number of walks of length one starting at $\mathbf{v}_i$ and ending at $\mathbf{v}_j$

Adjacency Matrix Theorem

A



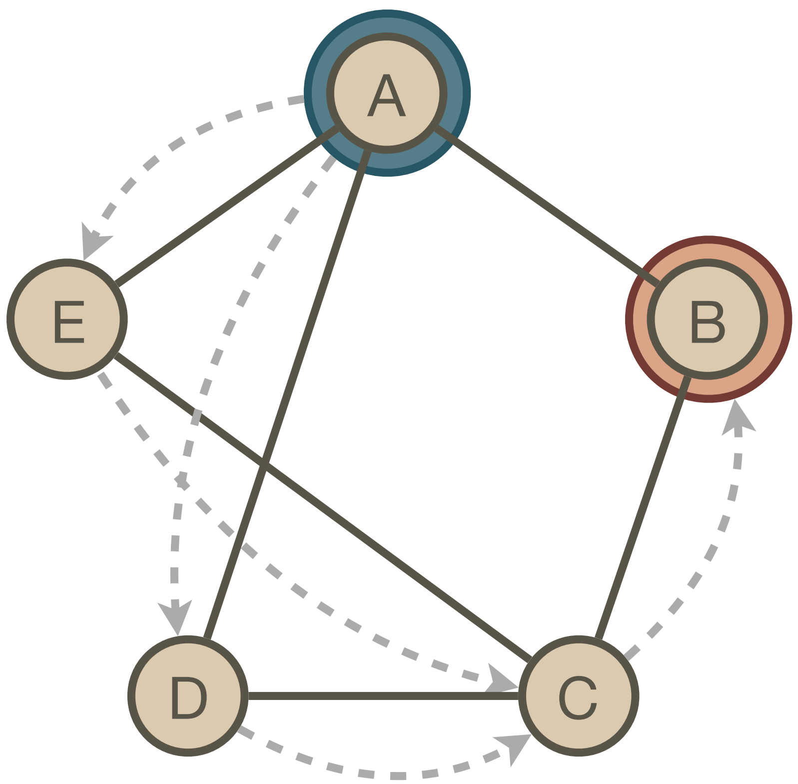
BA
EA
DA

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} * \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 & 0 & 0 \\ 0 & 2 & 0 & 2 & 2 \\ 3 & 0 & 3 & 0 & 0 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 & 2 \end{bmatrix}$$

The number of walks of length two starting at $\mathbf{v}_i$ and ending at $\mathbf{v}_j$

Adjacency Matrix Theorem

$\mathbf{A} * \mathbf{A} = \mathbf{A}^2$



BAB DAB
ECB EAB
DCB BCB

$$\begin{matrix} 3 & 0 & 3 & 0 & 0 \\ 0 & 2 & 0 & 2 & 2 \\ 3 & 0 & 3 & 0 & 0 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 & 2 \end{matrix} * \begin{matrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{matrix} = \begin{matrix} 0 & 6 & 0 & 6 & 6 \\ 6 & 0 & 6 & 0 & 0 \\ 0 & 6 & 0 & 6 & 6 \\ 6 & 0 & 6 & 0 & 0 \\ 6 & 0 & 6 & 0 & 0 \end{matrix}$$

The number of walks of length three starting at $\mathbf{v}_i$ and ending at $\mathbf{v}_j$

Adjacency Matrix Theorem

$\mathbf{A}^*\mathbf{A}^*\mathbf{A} = \mathbf{A}^{2*}\mathbf{A} = \mathbf{A}^3$

Adjacency Matrix Thm. Proof

- $\mathbf{A}$ is our base case. The adjacency matrix is by definition all the walks of length one!

- We will assume true for the k case. Let $\mathbf{A}^k = [n_{ij}]$.

- Now we need to show that the k+1 case is true. Let $\mathbf{A}^{k+1} = [t_{ij}]$.

  - Every walk of length k+1 is a walk of length one from $\mathbf{v}_i$ to a vertex $\mathbf{v}_q$ and then a walk of length k from $\mathbf{v}_q$ to $\mathbf{v}_j$.

Adjacency Matrix Thm. Proof

- So t_{ij} = sum(a_{iq}*n_{qj}) for q=1 though *n*
- If a_{iq}=0 then there are no walks of length one from $\mathbf{v}_i$ to $\mathbf{v}_q$
- If n_{qj}=0 then there are no walks of length k from $\mathbf{v}_q$ to $\mathbf{v}_j$
- If a_{iq}=1 then there are n_{qj} walks of length k+1 from $\mathbf{v}_i$ to $\mathbf{v}_j$
- Matrix multiplication:

$$\begin{bmatrix} a_{i1} & a_{i2} & a_{i3} & a_{i4} & a_{i5} \end{bmatrix} * \begin{bmatrix} n_{1j} \\ n_{2j} \\ n_{3j} \\ n_{4j} \\ n_{5j} \end{bmatrix} = a_{i1}*n_{1j} + a_{i2}*n_{2j} + a_{i3}*n_{3j} + a_{i4}*n_{4j} + a_{i5}*n_{5j}$$

Connectivity

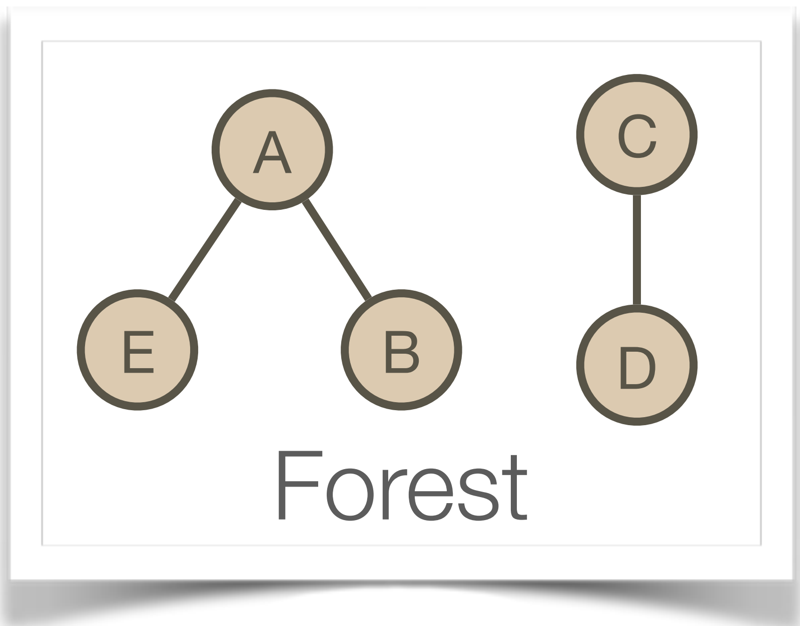
- Let $\mathbf{G}_1 = (\mathbf{V}_1, \mathbf{E}_1)$ and $\mathbf{G}_2 = (\mathbf{V}_2, \mathbf{E}_2)$ be graphs. The *union* of $\mathbf{G}_1$ and $\mathbf{G}_2$ is denoted by $\mathbf{G}_1 \cup \mathbf{G}_2$ and has a vertex set of $\mathbf{V}_1 \cup \mathbf{V}_2$ and and edge set of $\mathbf{E}_1 \cup \mathbf{E}_2$.

- A graph is *disconnected* if it is the union of two graphs.

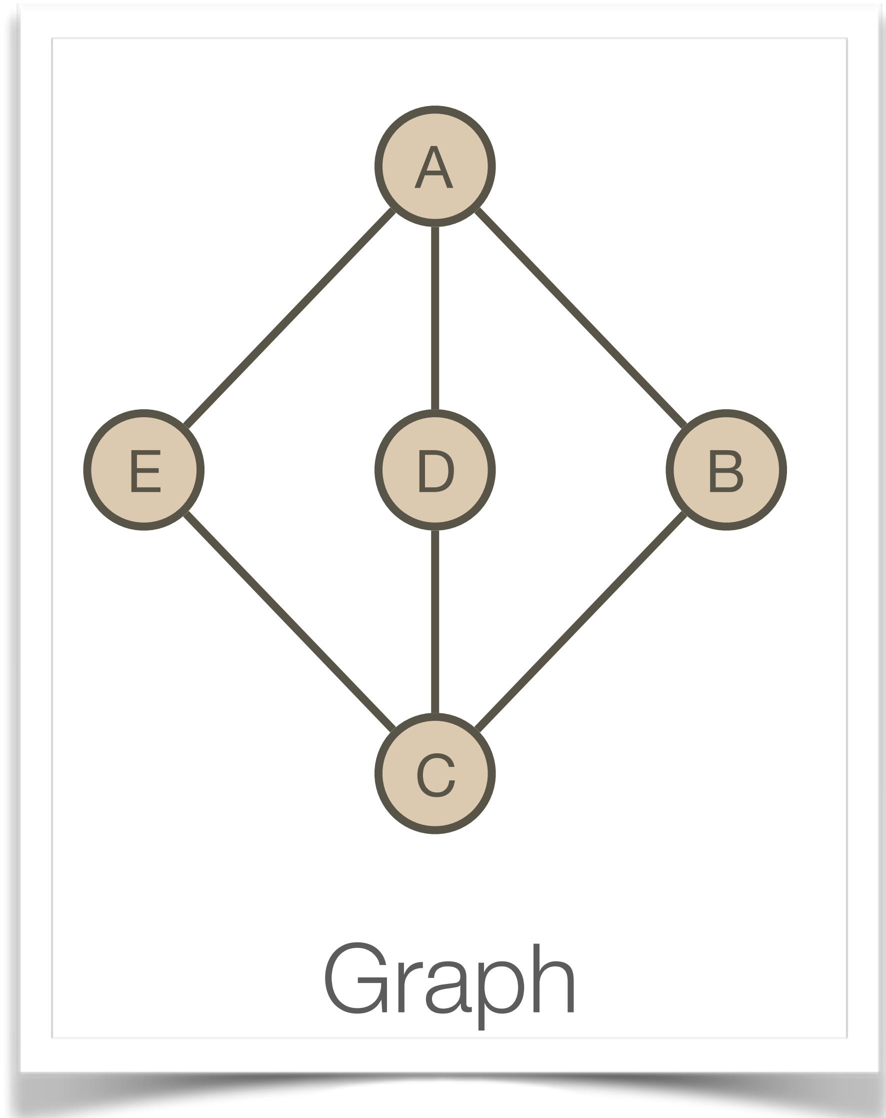
- A graph is *connected* if it is not disconnected.

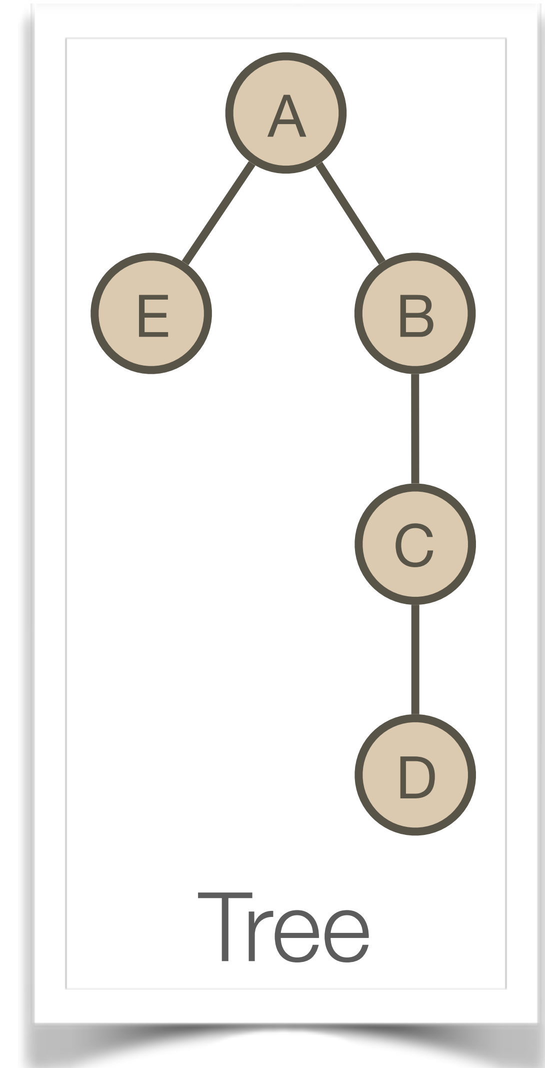
- A *component* of a graph is any of the connected graphs whose union forms the graph.

- The number of components of a graph $\mathbf{G}$ is $k(\mathbf{G})$



A *forest* is a graph with no cycles





A *tree* is a connected forest